Stability Analysis of 2-D PSV System Represented by Givone–Roesser Model

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Abstract
Multidimensional systems have been very interesting area from several past decades due to its increasing demands in various areas like digital signal processing, Image processing, Gas absorption, temperature measurements etc. As systems are multi-dimensional in nature stability of those systems decreases due to more than one dimension. These systems are difficult to represent, such systems can be represented by discrete state space models. There are several state space models are available among them FM1, FM2, GR, and R are very famous models even though they developed before four-five decade. In this paper our focus is to derive stability conditions for two-dimensional (2-D) Periodically Shift Varying (PSV) filters using GR models. As PSV filters has applicable in various places like Image Scrambling, video scrambling, design multiplier less filters etc. To check whether PSV system is stable or not, we have derived two sufficient stability conditions for PSV filter which helps to tell about stability. After that using Matlab LMI tool box we have done various analysis also An algorithm using linear matrix inequality is proposed to determine the stability of a given 2-D PSV system. As well as examples and comparative analysis also discussed in terms of stability for GR model.

Keywords
2-D Periodically Shift Varying Filters, Givone–Roesser Model, Stability, Linear Matrix inequality.

I. Introduction
Periodic shift varying systems have received wide spread attention over last three four decades due to its wide range applications as filtering of cyclostationary signals, Image scrambling, Speech scrambling, multi rate filter banks etc. [1]. Implementations of 2-D PSV filters compare to 1-D filters [2] is complicated due to its multidimensional nature. PSV filters are also important for designing 2-D filters with power of two coefficient [3]. That multipliers help to do real time processing with large amount of data. PSV systems have been analysed and designed from several past decades. [4] Which has direct form 2-D PSV filter have been analysed, equivalent shift invariant structure were derived in [5]. In [6] different formulation of two-dimensional filters with state space models which have been studied by several authors due to its efficient applicability. There are several authors who derived state space models for multi-dimensional systems but among them, The Givone-Roesser [7] and Fornasini-Marchesini [8] models are two such variations that have frequently been the subject of analysis. These models have developed for different purposes FMs models were developed for filtering purpose where GR model was developed for ILC (Iterative Learning Control) systems which are periodically repeated. Any filter form raises questions regarding stability, and these models are no exception. The objective of this paper is to establish two stability conditions for the Givone-Roesser model when they are shift variant. For 1-D state space models, stability conditions for time-varying systems have been derived in [9]. The condition is shown in [9] to be sufficient for stability of the system, and this result is the major building block on which the research herein is based.

Stability analysis of 2-D system are technically complicated than 1-D systems. Taking about shift-invariant difference equations form, the issue of a nonessential singularity of the second kind results in that there is no necessary and sufficient conditions for 2-D rational polynomials. Stability of not only 2-D but also multidimensional shift varying systems takes attentions [10] the stability of periodic 2-D system has also been studied [11]. Stability condition which is presented in [12] was studied. [13] which gives idea of asymptotic stability for two-dimensional shift variant system which is discrete in nature. [14] gives idea about robust stability for 2-D shift varying systems, [15] establishing stability in the case of constrained perturbations about a stable, shift-invariant Givone-Roesser model, and [16] presenting a very elegant result for the case where the system is positive. The goal of this paper is to derive sufficient conditions for PSV system using GR model.

II. GIVONE–ROESSER Model
The 2-D PSV system can be written in form of difference equation

\[ y(i,j) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn}(i,j)y(i-m,j-n) \]
\[ - \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} b_{mn}(i,j)u(i-m,j-n) \]  

(1)

The coefficient \( a_{mn}, b_{mn} \), with \((m, n) \neq (0,0)\) are used for are periodically shift variant system with periods (P,Q).

As with a 2-D LSIV system, several different state-space forms with periodic coefficient matrices can be used to represent the above 2-D PSV system. The Givone–Roesser (GR) model from (1) can be changed to

\[ \begin{pmatrix} x^h(i+1,j) \\ x^v(i+1,j) \end{pmatrix} = \begin{pmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{pmatrix} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + \begin{pmatrix} B_1(i,j) \\ B_2(i,j) \end{pmatrix} u(i,j) \]

(2)

In this paper, the zero-input stability of a GR state-space model is studied so that only the periodic coefficient matrices \( A_1(i,j), A_2(i,j), A_3(i,j), A_4(i,j) \) need to be reconsidered. A zero-input 2-D PSV GR model can be written as

\[ \begin{pmatrix} x^h(i+1,j) \\ x^v(i+1,j) \end{pmatrix} = \begin{pmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{pmatrix} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} \]

(3)

Note that the GR model can also be written in another form using the second model of FM [17] as follows. Define

\[ w(i,j) = \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} \]

(4)
Then the equation (4) can be modified as

\[ w(i,j) = A^{10}(i-1,j)w(i-1,j) + A^{01}(i,j-1)w(i,j-1) \]

\[ A^{10}(i,j) = \begin{pmatrix} A_1(i,j) & A_2(i,j) \\ 0 & 0 \end{pmatrix} \]

\[ A^{01}(i,j) = \begin{pmatrix} 0 & A_3(i,j) \\ A_5(i,j) & A_4(i,j) \end{pmatrix} \]

Where \( A^{10}(i,j) \) and \( A^{01}(i,j) \) are periodic with period \((P,Q)\) and \(O\) denotes a zero matrix of appropriate size for the application. The initial conditions can be assumed such that \( w(i,j) \), \( i < 0 \) or \( j < 0 \) and \( w(i,0) = w(0,j) \) for \( i \geq I, j \geq J \).

Stability conditions for the stability of 2-D PSV GR models are established using both absolute-value and vector-norm energy functions. Using matrix norms, we derive upper bounds of matrices given in theorem below.

**Theorem-1:** [Existing Result] [6] If

\[
\text{p}(\overline{T}_p) \prod_{r=0}^{r=S-1} |G_r| < 1
\]

where \( G_r \) [for \( r = 0, 1, ..., S-1 \)], \( \overline{T}_p \) and all relevant parameters are defined in [6], then

\[ \lim_{i+j \to \infty} w(i,j) = 0. \]

**III. Proposed Criteria**

As 2-D system has more than one parameter we can define them in by two points those are \( P_1 \) and \( P_2 \) where \( P_1 = (i_1,j_1) \) and \( P_2 = (i_2,j_2) \). This assumption is for representation of 2-D systems, so we can say that if \( P_1 > P_2 \) \( P_1 \) comes after \( P_2 \) while processing sequence or scan the order, \( P + 1 \) is indication of point that would processed after \( P \), normally \((i+1,j)\), unless a boundary in the space has been reached, and similarly for \( P - 1 \).

Sufficient condition for LSV 2-D GR is

\[
\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} - A^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} A > 0
\]

At the time of processing in Two dimensional system best and easiest way is first travers completely through one dimension then go through other dimensions in increasing order as well as process through next row of the first dimension. This approach is applied to a Givone-Roesser–type model, it implies that \( x^t \) is being updated at every iteration, and that there are constantly \( M \) versions of \( x^t \) kept in memory.

**Sufficient condition 1:**

For Stability analysis, Matrix \( A \) is change of \( A + \Delta A \). \( \Delta A \) is modelled into \( H, F, E \) to be able to use the Lemma. Derivation for second condition is more complex and hence few steps are described there.

Here, derivation is in the similar lines and hence result is shown directly. Lemma-1 used for derivations and Theorem-2 [Proposed result 1] is mentioned below.

Linear matrix inequality which helps to solve various problems in simple manner, in that Lyapunov equation are very famous in terms of stability due to its regional pole placements. Combining these specifications defines a multi-objective design problem that can be solved numerically.

**Lemma-1:**

Let \( A \in \mathbb{R}^{m \times m}, H \in \mathbb{R}^{m \times m}, E \in \mathbb{R}^{m \times m} \) and \( Q = Q^T \in \mathbb{R}^{m \times m} \) be the given matrices. Then, there exists a positive definite matrix \( P \) such that,

\[
(A + HFE)^T P (A + HFE) - Q < 0
\]

For all \( F \) satisfying \( F^T F \leq I \), if & only if there exists a scalar \( k > 0 \) such that

\[
\begin{pmatrix} P^{-1} + kHH^T \\ A^T \\ k^{-1}E^T E - Q \end{pmatrix} < 0
\]

**Theorem-2 [Proposed result 1]**

For 2-D PSV system (3) represented by GR models globally asymptotically stable provided there exists \( n \times n \) positive definite symmetric matrix \( P \) and positive scalar \( k \) such that,

\[
\begin{pmatrix} P^{-1} P A & 0 & PH \\ A^T P & -P & kE^T \\ 0 & kE & -kl \end{pmatrix} > 0
\]

Where Matrices H and E are

\[
\begin{pmatrix} H_1 & H_2 & 0 & 0 \\ 0 & H_3 & H_4 \\ E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}
\]

and \( A = (A_1, A_2, A_3, A_4) \)

**Sufficient Condition 2:**

For Stability analysis, Matrix \( A \) is change of \( A + \Delta A \). \( \Delta A \) is modelled into change of \( H, F, E \). Lemma-2 used for derivation followed by two important intermediate steps and Theorem-3 [Proposed result 2] are shown below:

**Lemma-2:**

Let represent change for corresponding and be real matrices of appropriate dimension with then the Lemma.

Let \( H, F, E \) represent change for corresponding and \( M \) be real matrices of appropriate dimension with \( M = M^T \) then the Lemma

\[
M + HFE + F^T E^T H^T < 0
\]

For all \( F^T F \leq I \) if and only if there exists scalar \( k > 0 \) such that

\[
M + kHH^T + k^{-1}E^T E < 0
\]

**1.**

\[
\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} - A^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} A + \begin{pmatrix} A_1 + H_1 F_1 E_1 & \cdots & A_1 + H_1 F_1 E_4 \\ \cdots & \cdots & \cdots \\ A_4 + H_4 F_4 E_1 & \cdots & A_4 + H_4 F_4 E_4 \end{pmatrix} > 0
\]

**2.**

\[
\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} - A^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} A + \begin{pmatrix} A_1 + H_1 F_1 E_1 & \cdots & A_1 + H_1 F_1 E_4 \\ \cdots & \cdots & \cdots \\ A_4 + H_4 F_4 E_1 & \cdots & A_4 + H_4 F_4 E_4 \end{pmatrix} > 0
\]

\[
\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} - \begin{pmatrix} A_1^T & \cdots & A_4^T \\ \cdots & \cdots & \cdots \\ A_4^T & \cdots & A_1^T \end{pmatrix} + \begin{pmatrix} (H_1 F_1 E_1)^T & \cdots & (H_1 F_1 E_4)^T \\ \cdots & \cdots & \cdots \\ (H_4 F_4 E_1)^T & \cdots & (H_1 F_1 E_4)^T \end{pmatrix} > 0
\]

\[
\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} - \begin{pmatrix} A_1 & \cdots & A_4 \\ \cdots & \cdots & \cdots \\ A_4 & \cdots & A_1 \end{pmatrix} + \begin{pmatrix} (H_1 F_1 E_1) & \cdots & (H_1 F_1 E_4) \\ \cdots & \cdots & \cdots \\ (H_4 F_4 E_1) & \cdots & (H_1 F_1 E_4) \end{pmatrix} > 0
\]
3. Theorem-3 [Proposed result 2]
For 2-D PSV system (3) represented by GR model is globally asymptotically stable provided that, there exists $n \times n$ positive definite symmetric matrices $P_1, P_2$ and positive scalars $k_1, k_2, k_3, k_4$ such that $S > 0$.

Where $S$ is defined below.

\[
S = \begin{pmatrix}
0 & \ldots & 0 & -k_1E & -k_1E & 0 \\
0 & \ldots & 0 & -k_2A_2 & 0 & 0 \\
0 & \ldots & 0 & -k_2A_2 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & -k_3E & -k_3E & 0 \\
0 & \ldots & 0 & -k_3E & -k_3E & 0 \\
0 & \ldots & 0 & -k_3E & -k_3E & 0 \\
\end{pmatrix}
\]

Comment:
As we have two results for stability analysis of 2-D PSV systems, we need to take decision on which of these criterion is to be used for stability analysis. So, the proposed flow chart of the proposed work with two criteria can be shown as follows.

By using the above flow chart, the two step stability analysis criteria can be implemented. As they are LMI based criteria, it has two implicit advantages:
1. They are implementable by tools such as, MATLAB LMI TOOL BOX.
2. By hundreds of LMI based results existing in literature, it is evident that LMI based stability criteria are more relaxed. That means, in 2-d case there cannot be necessary and sufficient condition. But, Gap between them may have been minimized as compared to the results existing in literature [6]. Of course, this claim need to be validated by numerical example as done in [18] for similar results of the system represented by FM-1.

Taking [19]'s example as references and comparison of our result with existing results for stability.

Table 1 [19].

<table>
<thead>
<tr>
<th>Ex/Thm</th>
<th>Crit 1 suff cond</th>
<th>Thm 1 suff cond</th>
<th>Thm 2 suff cond</th>
<th>Thm 4 acc cond</th>
<th>Thm 3 suff cond</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.83</td>
<td>3.83</td>
<td>3.83</td>
<td>3.83</td>
<td>3.83</td>
<td>Stable</td>
</tr>
<tr>
<td>2</td>
<td>1.10 (1.10)</td>
<td>1.10 (1.10)</td>
<td>1.10 (1.10)</td>
<td>1.10 (1.10)</td>
<td>1.10 (1.10)</td>
<td>Stable</td>
</tr>
<tr>
<td>3</td>
<td>1.17 (1.17)</td>
<td>1.17 (1.17)</td>
<td>1.17 (1.17)</td>
<td>1.17 (1.17)</td>
<td>1.17 (1.17)</td>
<td>Stable</td>
</tr>
<tr>
<td>4</td>
<td>1.37 (1.37)</td>
<td>1.37 (1.37)</td>
<td>1.37 (1.37)</td>
<td>1.37 (1.37)</td>
<td>1.37 (1.37)</td>
<td>Stable</td>
</tr>
<tr>
<td>5</td>
<td>3.60 (3.60)</td>
<td>3.60 (3.60)</td>
<td>3.60 (3.60)</td>
<td>3.60 (3.60)</td>
<td>3.60 (3.60)</td>
<td>Stable</td>
</tr>
<tr>
<td>6</td>
<td>2.89 (2.89)</td>
<td>2.89 (2.89)</td>
<td>2.89 (2.89)</td>
<td>2.89 (2.89)</td>
<td>2.89 (2.89)</td>
<td>Stable</td>
</tr>
</tbody>
</table>

The result we have obtained

Table 2: Our results

<table>
<thead>
<tr>
<th>Ex/Thm</th>
<th>Thm 1</th>
<th>Thm 2</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>√</td>
<td>-</td>
<td>stable</td>
</tr>
<tr>
<td>2</td>
<td>×</td>
<td>√</td>
<td>stable</td>
</tr>
<tr>
<td>3</td>
<td>×</td>
<td>√</td>
<td>stable</td>
</tr>
<tr>
<td>4</td>
<td>√</td>
<td>-</td>
<td>stable</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
<td>unstable</td>
</tr>
<tr>
<td>6</td>
<td>×</td>
<td>√</td>
<td>stable</td>
</tr>
</tbody>
</table>

Table 1 is analysis table of examples which has been concluded in the [19]. We have taken same examples and performed for our derived stability theorem 1 and 2, analysis for that is available in Table 2, in that we conclude that our stability criteria has only two theorem which can easily give stability information about system. More ever in terms of computational complexity factor and other parameters is discussed in Table 3. Table 3 gives over all analysis of our derived criteria.
Table 3: (Stability Analysis of 2D PSV System for GR model)

<table>
<thead>
<tr>
<th>Proposed Method(I)</th>
<th>Proposed Method(II)</th>
<th>Existing Methods</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stability Criteria</td>
<td>Stability Criteria</td>
<td>Total Six Result</td>
<td>Proposed method is LMI based, existing method is linear algebra based</td>
</tr>
<tr>
<td>Computational complexity, memory</td>
<td>Less</td>
<td>More</td>
<td>Much more(Total Six algorithm)</td>
</tr>
<tr>
<td>Speed</td>
<td>More</td>
<td>Less</td>
<td></td>
</tr>
<tr>
<td>Gap between necessary and sufficient Conditions</td>
<td>More</td>
<td>Less</td>
<td>Much more</td>
</tr>
<tr>
<td>Ease of handling algorithm for stability criteria</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Stability Analysis of real life example 1:

To do stability analysis of real life example we have taken example of 2-D LSV digital filter from [20] which can be written in form of difference equation is

\[ y(i+1,j+1)=Q[a(i,j)y(i+1,j)+b(i,j)y(i,j+1)]+U(i,j) \]  

Equation (7) can be represented as figure 1 below.

Equation (7) can be represented as figure 1 below.

where Q represents a quantization operation, U(i,j) is input which assumed as zero input (i,j) also b(i,j) taken as zero. Here,

\[ a(i,j) = \frac{1}{2}(\sin(2i+1) + \cos(2j + 1)) \]  

in this case a(i,j) periodic separable with period (2,2) and this system is stable according to [20]. It may be represented in matrix form as

\[ a(i,j) = \begin{pmatrix} a(0,0) & a(0,1) \\ a(1,0) & a(1,1) \end{pmatrix} = \begin{pmatrix} 0.51 & 0.51 \\ -0.54 & -0.53 \end{pmatrix} \]  

From the above equations we can write matrix for

\[ a(i+2,j) = \begin{pmatrix} 0.56 & 0.56 \\ -0.59 & -0.59 \end{pmatrix} \]  

\[ a(i,j+2) = \begin{pmatrix} 0.5 & 0.5 \\ -0.53 & -0.53 \end{pmatrix} \]  

Taking a(i,j)=A and a(i+2,j), a(i,j+2) are matrices which can be help for finding \( \Delta A \) Respected to equation (10), (11) we can write \( \Delta A \) for equation (9) is

\[ \Delta A = \begin{pmatrix} 0.05 & 0.05 \\ 0.05 & 0.06 \end{pmatrix} \]  

Using this \( \Delta A \), \( \Delta A = HFE \) we can find HFE and follow the steps which discussed in [18]. As \( FT \leq 1 \) for that we have taken

\[ F = \begin{pmatrix} 0.95 & 0 \\ 0 & 0.95 \end{pmatrix} \]  

\[ H = \begin{pmatrix} 0.31 & 0.1 & 0 \\ 0 & 0 & 0.23 \end{pmatrix} \]  

\[ E = \begin{pmatrix} -0.03 & 0.2 \\ 0 & 0 \\ -0.04 & 0.3 \end{pmatrix} \]  

\[ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  

Using above values in Matlab LMI toolbox for equation (5) which is our first proposed criteria. Matlab LMI toolbox gives \( t=-0.14 \) which is a negative value. According to Matlab LMI toolbox, negative value suggests that system is feasible. Same procedure we can follow for b (i, j) and checked its stability. We have already taken the stable system as discussed previously. Here we get stability in our first criteria only so we do not have to go through second criteria, as we have discussed in the derived flow chart for stability previously.
Example 2:
We have borrowed example from [21] and taken it for e.q (2) where the coefficient are periodic with period and given above.

\[
A(0,0) = \begin{pmatrix} 0 & 0.2 \\ 0 & 0 \end{pmatrix}, \quad A(1,0) = \begin{pmatrix} 0.3 & 0 \\ 0.1 & -0.1 \end{pmatrix}
\]
\[
A(0,1) = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad A(1,1) = \begin{pmatrix} -0.18 & 0.2 \\ 0 & 0.05 \end{pmatrix}
\]

Using \( A(0,0) = A \) and \( A(0,1), A(1,0), A(1,1) \) used for find \( \Delta A \), we have followed steps which are discussed in [18]. As \( F^T F \leq I \) for that we have taken.

\[
\Delta A = \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.31 \end{pmatrix}
\]
\[
F = \begin{pmatrix} 0.95 \\ 0 \end{pmatrix}
\]
\[
H = \begin{pmatrix} 0.31 & 0 & 0 \\ 0 & 0 & 0.23 & 0.2 \end{pmatrix}
\]
\[
E = \begin{pmatrix} 0.34 & 0.77 \\ 0 & 0 \\ 0 & 0 \\ -1.24 & 2.09 \end{pmatrix}
\]
\[
i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Using above values in Matlab LMI toolbox for equation (5) which is our first proposed criteria. Matlab LMI toolbox gives \( t = -0.013 \) so system is feasible, which suggest the system is stable. Here also we get stability in our first criteria only so following our derived flow chart we do not go towards second derived criteria.

**IV. Conclusion**

In this paper, we have presented two important results for the Givone-Roesser model of 2-D PSV system. Two sufficient conditions are established for the stability which can be done using Lyapunov inequality is an LMI expressing stability criteria. So our first condition for stability is relaxed as far as Computational complexity is concerned and second condition could be applied if first condition for stability fails. Hence this makes the algorithm more robust and useful for the stability analysis of GR model systems. The supremacy of the algorithm is verified with numerical example as well as comparisons with existing results.

**References**


Prashant K. Shah received the B.E degree from the M.S University, Baroda in 1991 after completing it he continued the M.E with specialization in ‘Microprocessor Systems and Applications’. From 1992 to 1997 he was working as a lecturer, he joined the S.V. National Institute of Technology, Surat in 1998, where he is currently an Associate Professor. His current research interests are in the areas of Signal Processing, Estimation and filtering, PSV filtering and its stability analysis.